Notes on other stuff

Some author

July 7, 2024

Contents

1 Splines

Shchepetkin and McWilliams (2005) consider a piecewise-parabolic reconstruction of the vertical density profile from a set of discrete values $\{\overline{\rho}_k | k = 1, ..., N\}$ that is interpreted as a set of grid-box averages within each vertical vertical grid box H_k,

$$\rho(z') = \overline{\rho}_k + \frac{\rho_{k+\frac{1}{2}} - \rho_{k-\frac{1}{2}}}{H_k} z' + 6 \left(\frac{\rho_{k+\frac{1}{2}} + \rho_{k-\frac{1}{2}}}{2} - \overline{\rho}_k \right) \left[\frac{{z'}^2}{H_k^2} - \frac{1}{12} \right]. \tag{1}$$

Eq. (1) is similar to Eq. 1.4 of Colella and Woodward (1984) but uses a non-normalized local vertical coordinate z' such that $-\frac{H_k}{2} \leq z' \leq \frac{H_k}{2}$ and

$$\begin{aligned} \rho(\frac{H_k}{2}) &= \rho_{k+\frac{1}{2}} \\ \rho(-\frac{H_k}{2}) &= \rho_{k-\frac{1}{2}} \end{aligned} \tag{2}$$

are the density values at the upper and lower grid box interfaces, respectively. Note that

$$\frac{1}{H_k} \int_{-\frac{H_k}{2}}^{+\frac{H_k}{2}} \rho(z')\, dz' = \overline{\rho}_k, \tag{3}$$

and in particular that the integral Eq. (3) is independent of the values $\rho_{k\pm\frac{1}{2}}$.

We try to reproduce the algorithms to compute

1. a reconstruction of the density at the grid box interfaces
2. a reconstruction of the vertical density derivatives at the grid box interfaces, similar to what is done in the function `s_balance.m` for temperature/salinity.

1.1 Reconstruction of density

First, the values of density are reconstructed at the interfaces. Given $\{\overline{\rho}_k|k=1,...,N\}$ there are $2N$ unknown interface values, but the notation Eq. (1) already implicitly postulates the continuity of the vertical density distribution, i.e.

$$\rho(\frac{H_k}{2}) = \rho(-\frac{H_{k+1}}{2}) \qquad k=1,...,N-1, \tag{4}$$

because both sides of Eq. (4) are denoted using the same expression $\rho_{k+\frac{1}{2}}$. The notation implicitly accounts for $N-1$ equations expressing continuity, leaving $N+1$ unknown interface values $\{\rho_{k+\frac{1}{2}}|k=0,...,N\}$. Continuity of the vertical derivative of the density distribution implies

$$\rho'(\frac{H_k}{2}) = \rho'(-\frac{H_{k+1}}{2}) \qquad k=1,...,N-1, \tag{5}$$

where $\rho'(z')$ is from Eq. (1)

$$\rho'(z') = \frac{\rho_{k+\frac{1}{2}} - \rho_{k-\frac{1}{2}}}{H_k} + \frac{12}{H_k^2}\left(\frac{\rho_{k+\frac{1}{2}} + \rho_{k-\frac{1}{2}}}{2} - \overline{\rho}_k\right) z', \tag{6}$$

yielding another $N-1$ equations, leaving 2 equations e.g. for prescription of the first derivative at the bottom and the surface. Since

$$\rho'(\frac{H_k}{2}) = \frac{1}{H_k}\left(4\rho_{k+\frac{1}{2}} + 2\rho_{k-\frac{1}{2}} - 6\overline{\rho}_k\right) \tag{7}$$

$$\rho'(-\frac{H_k}{2}) = \frac{1}{H_k}\left(-2\rho_{k+\frac{1}{2}} - 4\rho_{k-\frac{1}{2}} + 6\overline{\rho}_k\right), \tag{8}$$

(9)

each equation of (5) can be written as

$$\frac{2}{H_k}\rho_{k-\frac{1}{2}} + 4\left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\rho_{k+\frac{1}{2}} + \frac{2}{H_{k+1}}\rho_{k+\frac{3}{2}} = 6\left(\frac{\overline{\rho}_k}{H_k} + \frac{\overline{\rho}_{k+1}}{H_{k+1}}\right). \quad (10)$$

Choosing the boundary conditions $\rho'(-\frac{H_1}{2}) = \rho'(\frac{H_N}{2}) = 0$ yields

$$\frac{4}{H_1}\rho_{\frac{1}{2}} + \frac{2}{H_1}\rho_{\frac{3}{2}} = \frac{6}{H_1}\overline{\rho}_1 \quad (11)$$

$$\frac{2}{H_N}\rho_{N-\frac{1}{2}} + \frac{4}{H_N}\rho_{N+\frac{1}{2}} = \frac{6}{H_N}\overline{\rho}_N \quad (12)$$

$$(13)$$

and Eqs. (11), (12) and (10) form a linear tridiagonal system of $N+1$ equations for the interface values $\{\rho_{k+\frac{1}{2}}|k = 0, \dots, N\}$.

1.2 Reconstruction of the vertical derivative

Second, from Eqs. (5) follows via (7) and (8)

$$\rho'(\frac{H_k}{2})\,(H_k + H_{k+1}) = 2\left(\rho_{k-\frac{1}{2}} - \rho_{k+\frac{3}{2}}\right) + 6\left(\overline{\rho}_{k+1} - \overline{\rho}_k\right) \quad (14)$$

$$H_k\rho'\left(-\frac{H_k}{2}\right) + H_{k+1}\rho'\left(\frac{H_{k+1}}{2}\right) = 4\left(\rho_{k+\frac{3}{2}} - \rho_{k-\frac{1}{2}}\right) + 6\left(\overline{\rho}_k - \overline{\rho}_{k+1}\right) \quad (15)$$

Multiplying Eq. (14) by two and adding (15) yields

$$H_k\rho'\left(-\frac{H_k}{2}\right) + 2\rho'(\frac{H_k}{2})\,(H_k + H_{k+1}) + H_{k+1}\rho'\left(\frac{H_{k+1}}{2}\right) = 6\left(\overline{\rho}_{k+1} - \overline{\rho}_k\right) \quad (16)$$

$$(17)$$

Denoting

$$\rho'_{k-\frac{1}{2}} := \rho'\left(-\frac{H_k}{2}\right) \quad (18)$$

$$\rho'_{k+\frac{1}{2}} := \rho'\left(\frac{H_k}{2}\right) \quad (19)$$

$$\rho'_{k+\frac{3}{2}} := \rho'\left(\frac{H_{k+1}}{2}\right), \quad (20)$$

$$(21)$$

Eq. (16) becomes

$$H_k\rho'_{k-\frac{1}{2}} + 2\left(H_k + H_{k+1}\right)\rho'_{k+\frac{1}{2}} + H_{k+1}\rho'_{k+\frac{3}{2}} = 6\left(\overline{\rho}_{k+1} - \overline{\rho}_k\right), \tag{22}$$

Choosing again $\rho'_{\frac{1}{2}} = \rho'_{N+\frac{1}{2}} = 0$ yields a tridiagonal linear system with $N-1$ equations for $\{\rho'_{k+\frac{1}{2}} | k = 1, \ldots, N-1\}$. The matrix is

$$\mathbf{A} := \begin{pmatrix} 2(H_1+H_2) & H_2 & & & 0 \\ H_2 & 2(H_2+H_3) & H_3 & & \\ & \ddots & \ddots & \ddots & \\ & & H_{N-2} & 2(H_{N-2}+H_{N-1}) & H_{N-1} \\ 0 & & & H_{N-1} & 2(H_{N-1}+H_N) \end{pmatrix} \tag{23}$$

Contents

References

Colella, P. and P. R. Woodward (1984). *The piecewise parabolic method (PPM) for gas-dynamical simulations.* URL: `https://www.sciencedirect.com/science/article/pii/0021999184901438`.

Shchepetkin, A.F. and J.C. McWilliams (2005). *The regional oceanic modeling system (ROMS): a split-explicit, free-surface, topography-following-coordinate oceanic model*. URL: `https://doi.org/10.1016/j.ocemod.2004.08.002`.